

# A closed form to the general solution of linear difference equations with variable coefficients

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## Abstract

The determinant of a lower Hessenberg matrix (Hessenbergian) is expressed as a sum of signed elementary products indexed by initial segments of nonnegative integers. A closed form alternative to the recurrence expression of Hessenbergians is thus obtained. This result further leads to a closed form of the general solution for regular order linear difference equations with variable coefficients, including equations of  $N$  order and equations of ascending order.

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## 1 Introduction

Higher order linear difference equations with time varying coefficients (LDEVCs) and their solutions come to focus, because of their ability to capture and model the dynamics of natural and social phenomena including abrupt and structural changes. An explicit expression for the general solution of the second order LDEVC was presented by Popenda in [1]. A representation of their general solution in terms of a single matrix determinant was established by Kittappa in [2]. Closed form solutions for homogeneous LDEVC of order  $N \geq 2$  have been presented by Mallik in [3, 4], who also provides, in [5], an explicit expression for the general solution of the non-homogeneous case. Despite their theoretical significance such solution expressions have not been utilized in scientific modelling. A closed form for the general solution of LDEVC of order greater than 1 is a long-standing problem (see [6]).

It has been established, in [7], that the infinite Gauss-Jordan algorithm under a rightmost pivot elimination strategy constructs the general solution sequence of row-finite systems. This type of infinite linear systems was utilized to represent LDEVCs of regular and irregular order. Furthermore, the class of ascending order LDEVCs has been introduced in order to extend the class of  $N$ th order equations to cover regular order LDEVCs. It has been shown that the application of the infinite Gaussian elimination algorithm to a LDEVC of regular order generates solutions (general homogeneous and non-homogeneous) in terms of Hessenbergians. Applying the solution formula to the first order LDEVC, the well known closed form solution (see [6]) is recovered. Applying the same formula to the  $N$ th order LDEVC, the general solution obtained in [2] is also recovered.

In this paper, we present an alternative expression to the recurrence formula (see (11)) for the  $n$ th order Hessenbergian, in closed form (see formula (26)). Unlike in the Leibniz formula for determinants, which consists of  $n!$  signed elementary products (SEPs) and in which the sum variable ranges over the symmetric group of permutations, the expression obtained here is a sum of  $2^{n-1}$  (non-trivial) SEPs in which the sum variable ranges over the set of integers in the initial segment  $[0, 2^{n-1} - 1]$ . This is due to an expression of each SEP as an image of a composite  $\chi^{(n)}$  of two bijections,  $\varphi^{(n)}$  and  $\tau^{(n)}$ . More specifically, we take advantage of the special structure of non-trivial SEPs associated with Hessenbergians (see section 3) to obtain (in section 4) a direct representation of these SEPs as  $n$ -dimensional arrays of

0s and 1s through a bijection  $f^{(n)}$ . The set of such arrays can be viewed as the set consisting of binary representations of the integers in  $[0, 2^{n-1} - 1]$ , denoted by  $\mathcal{B}_{n-1}$ . The function  $\varphi^{(n)}$  stands for the inverse of  $f^{(n)}$ , which maps each  $r \in \mathcal{B}_{n-1}$  to the SEP  $\varphi^{(n)}(r)$ . The bijection  $\tau^{(n)}$  maps integers from  $[0, 2^{n-1} - 1]$  to binaries in  $\mathcal{B}_{n-1}$  and is described in terms of elementary integer functions involving the greatest and the modulo function (see (25)).

Instead of applying the standard transformation of the original LDEVC into another difference equation with new coefficients, as followed in [4], we use the complete lower Hessenberg form of the solution matrices associated with the ascending order LDEVC. We thus obtain a closed form to the general solution for LDEVCs (see formula (28)) through the closed form expression of Hessenbergians. The general solution of the  $N$ th order LDEVC is included as a special case.

The general solution of the  $N$ th order LDEVC and its closed form leads to the development of a unified theory for time series models with varying coefficients as established in [8]. An application of this theory is the modelling of stock volatilities during financial crises as presented in [9]. Another application is the parsimonious formulation of periodic ARMA models [10].

## 2 Hessenbergian solutions of linear difference equations

Let  $\mathbb{Z}$  (resp.  $\mathbb{Z}^*$ ,  $\mathbb{Z}^+$ ) be the set of integers (resp. non-negative integers, positive integers) and  $\mathbb{C}$  be the algebraic field of complex numbers. The linear difference equation with variable coefficients (LDEVC) is defined by the recurrence

$$a_{n,0}y_{-N} + a_{n,1}y_{1-N} + \dots + a_{n,N}y_0 + a_{n,N+1}y_1 + \dots + a_{n,N+n-1}y_{n-1} + a_{n,N+n}y_n = g_n, \quad n \in \mathbb{Z}^*, \quad (1)$$

where  $N$  is a fixed non-negative integer and  $a_{n,i}, g_n \in \mathbb{C}$  are values of arbitrary (complex valued) functions.

If  $a_{n,N+n} \neq 0$  for all  $n \in \mathbb{Z}^*$  and  $a_{m,0} \neq 0$  for some  $m \in \mathbb{Z}^*$  in (1), then the LDEVC is referred to as *ascending order linear difference equation* of index  $N$ . The sequence of equations in (1) can be written as an infinite  $\mathbb{Z}^* \times \mathbb{Z}^*$  linear system

$$\mathbf{A} \cdot \mathbf{y} = \mathbf{g}, \quad (2)$$

where

$$\mathbf{A} = \begin{pmatrix} a_{0,0} & a_{0,1} & \dots & a_{0,N-1} & a_{0,N} & 0 & \dots & 0 & 0 & \dots \\ a_{1,0} & a_{1,1} & \dots & a_{1,N-1} & a_{1,N} & a_{1,N+1} & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\ a_{n-1,0} & a_{n-1,1} & \dots & a_{n-1,N-1} & a_{n-1,N} & a_{n-1,N+1} & \dots & a_{n-1,N+n-1} & 0 & \dots \\ a_{n,0} & a_{n,1} & \dots & a_{n,N-1} & a_{n,N} & a_{n,N+1} & \dots & a_{n,N+n-1} & a_{n,N+n} & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (3)$$

$\mathbf{y} = (y_{-N}, y_{1-N}, \dots, y_{-1}, y_0, y_1, \dots)^T \in \mathbb{C}^\infty$  and  $\mathbf{g} = (g_0, g_1, g_2, \dots)^T \in \mathbb{C}^\infty$  (“ $T$ ” stands for transposition).

If  $a_{n,N+n} = 0$  for some  $n \in \mathbb{Z}^*$  and  $a_{m,N+m} \neq 0$  for some  $m \in \mathbb{Z}^*$  with  $m \neq n$  in (1), the row-lengths of  $\mathbf{A}$  can vary irregularly and (1) is referred to as *linear difference equation of irregular order*. Otherwise it would be referred to as *linear difference equation of regular order*. The general solution sequence of equations of irregular order is constructed by implementing the infinite Gauss-Jordan algorithm under a rightmost pivot elimination strategy (see [7]).

If  $a_{n,N+n} \neq 0$  for all  $n \in \mathbb{Z}^*$ ,  $a_{m,m} \neq 0$  for some  $m \in \mathbb{Z}^*$  and  $a_{n,i} = 0$  for all  $i, n \in \mathbb{Z}^*$  such that  $0 \leq i < n$ , then (1) turns into a *linear difference equation with variable coefficients of order  $N$* . Letting  $N = 0$  and  $a_{n,n} \neq 0$  for all  $n \in \mathbb{Z}^*$ , then (1) turns into a linear difference equation with variable coefficients of *unbounded order*, as named by Mallik in [5]. In the terminology used herein, equations of unbounded order can be described as equations of ascending order of index 0. Their matrix representation is non-singular, since it is lower triangular, with non-zero entries in the main diagonal. In this context, LDEVCs of ascending order and of constant order cover all LDEVCs of regular order. The most complete form of regular order LDEVCs is the ascending order one.

The coefficient matrix  $\mathbf{A}$  in (3) associated with a LDEVC of regular order is in lower echelon form. In this case, we solely implement the infinite Gaussian elimination. This yields a unique row equivalent matrix of  $\mathbf{A}$ , say  $\mathbf{H}$ , called Hermite form (HF) of  $\mathbf{A}$  (or lower row reduced echelon form of  $\mathbf{A}$ ). The first  $N$  opposite-sign columns of  $\mathbf{H}$  augmented at their top by  $N$  distinct unit vectors, turn out to be

linearly independent homogeneous solution sequences of the ascending order LDEVC. As a consequence, an ascending order LDEVC of index  $N$  has, as in the case of LDEVCs of  $N$ -order (see [6]),  $N$  linearly independent homogeneous solutions that span the space of homogeneous solutions, thus forming an algebraic basis of this space. This basis will be denoted by  $\xi = \{\xi_i, 0 \leq i \leq N-1\}$ . Formally  $\xi$  extends the notion of the *fundamental solution set* associated with the  $N$ -order LDEVC (see [6]).

For every  $i$  such that  $0 \leq i \leq N-1$  the fundamental solution matrix  $\Xi_n^{(i)}$  is the lower Hessenberg matrix:

$$\Xi_n^{(i)} = \begin{pmatrix} a_{0,i} & a_{0,N} & 0 & \dots & 0 \\ a_{1,i} & a_{1,N} & a_{1,N+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,i} & a_{n-1,N} & a_{n-1,N+1} & \dots & a_{n-1,N+n-1} \\ a_{n,i} & a_{n,N} & a_{n,N+1} & \dots & a_{n,N+n-1} \end{pmatrix}.$$

The set  $\xi$  of fundamental solution sequences consists of the sequences

$$\begin{aligned} \xi_0 &= (1, 0, \dots, 0, \quad \xi_{0,0}, \quad \xi_{1,0}, \quad \dots \quad \xi_{n,0}, \quad \dots)^T \\ \xi_1 &= (0, 1, \dots, 0, \quad \xi_{0,1}, \quad \xi_{1,1}, \quad \dots \quad \xi_{n,1}, \quad \dots)^T \\ &\vdots \\ \xi_{N-1} &= (0, 0, \dots, 1, \quad \xi_{0,N-1}, \quad \xi_{1,N-1}, \quad \dots \quad \xi_{n,N-1}, \quad \dots)^T, \end{aligned}$$

where the general term  $\xi_{n,i}$ , referred to as *fundamental solution*, is given by

$$\xi_{n,i} = (-1)^{n+1} \frac{\det(\Xi_n^{(i)})}{a_{0,N} \cdot a_{1,N+1} \dots a_{n,n+N}} \quad n \geq 0.$$

The general term of the particular solution sequence  $P = (0, 0, \dots, 0, p_0, p_1, \dots, p_n, \dots)^T$  of the ascending order LDEVC, referred to as *particular solution*, is given by

$$p_n = (-1)^n \frac{\det(\mathbf{P}_n)}{a_{0,N} \cdot a_{1,N+1} \dots a_{n,n+N}}, \quad n \geq 0, \quad (4)$$

where

$$\mathbf{P}_n = \begin{pmatrix} g_0 & a_{0,N} & 0 & \dots & 0 \\ g_1 & a_{1,N} & a_{1,N+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{n-1} & a_{n-1,N} & a_{n-1,N+1} & \dots & a_{n-1,N+n-1} \\ g_n & a_{n,N} & a_{n,N+1} & \dots & a_{n,N+n-1} \end{pmatrix}.$$

The general solution sequence of the ascending order LDEVC, as the sum of the homogeneous (a linear combination of fundamental solutions) and particular solutions, is given by

$$y = (y_{-N}, \dots, y_{-1}, p_0 + \sum_{k=0}^{N-1} \xi_{0,k} y_{k-N}, p_1 + \sum_{k=0}^{N-1} \xi_{1,k} y_{k-N}, \dots)^T, \quad (5)$$

where  $y_{-N}, \dots, y_{-1}$  are arbitrary constants. The *general solution matrix* associated with the ascending order LDEVC is given by the lower Hessenberg matrix

$$\mathbf{G}_n = \begin{pmatrix} g_0 - \sum_{k=0}^{N-1} a_{0,k} y_{k-N} & a_{0,N} & 0 & \dots & 0 \\ g_1 - \sum_{k=0}^{N-1} a_{1,k} y_{k-N} & a_{1,N} & a_{1,N+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{n-1} - \sum_{k=0}^{N-1} a_{n-1,k} y_{k-N} & a_{n-1,N} & a_{n-1,N+1} & \dots & a_{n-1,n+N-1} \\ g_n - \sum_{k=0}^{N-1} a_{n,k} y_{k-N} & a_{n,N} & a_{n,N+1} & \dots & a_{n,n+N-1} \end{pmatrix}. \quad (6)$$

As a result of the multilinear property of determinants, the  $n$ th (or general) term  $y_n = p_n + \sum_{k=0}^{N-1} \xi_{n,k} y_{k-N}$  of  $y$  in (5), referred to as *general solution*, is represented in terms of Hessenbergians as follows:

$$y_n = (-1)^n \frac{\det(\mathbf{G}_n)}{a_{0,N} \cdot a_{1,N+1} \dots a_{n,n+N}}. \quad (7)$$

A major objective of the present work is to provide a closed form expression for  $\det(\mathbf{G}_n)$ .

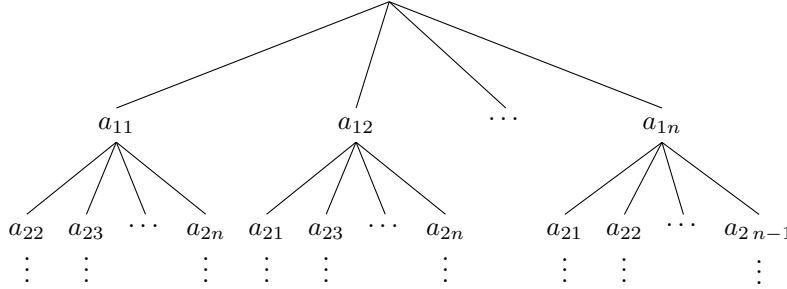
### 3 Hessenbergians and non-trivial signed elementary products

Let  $S_n$  be the group of permutations on  $\{1, 2, \dots, n\}$ , known as symmetric group of order  $n$ . The signature  $\text{sgn}(\ell)$  of  $\ell \in S_n$  is defined as  $-1$  if  $\ell$  is odd and  $+1$  if  $\ell$  is even. Let  $\ell \in S_n$ . A *signed elementary product* (SEP) of a square matrix  $\mathbf{A} = (a_{i,j})_{1 \leq i,j \leq n}$  over  $\mathbb{C}$  is an ordered pair  $(\ell, \text{sgn}(\ell) \prod_{i=1}^n a_{i,\ell_i})$ , that is an element of  $S_n \times \mathbb{C}$ . The second component of a SEP is its *product value* in  $\mathbb{C}$ . We infer that two SEPs  $(\ell, \text{sgn}(\ell) \prod_{i=1}^n a_{i,\ell_i})$  and  $(l, \text{sgn}(l) \prod_{i=1}^n a_{i,l_i})$  of  $\mathbf{A}$  are equal if and only if  $\ell = l$ . Bearing this fact in mind, we shall use the standard notation of SEPs:  $\text{sgn}(\ell) a_{1,\ell_1} a_{2,\ell_2} \dots a_{n,\ell_n}$ ,  $\ell \in S_n$ . The set of SEPs of  $\mathbf{A}$  is in one-to-one correspondence with  $S_n$ . Therefore the number of all SEPs of  $\mathbf{A}$  coincides with the number of all permutations on  $n$  objects, that is  $\text{card}(S_n) = n!$ . The determinant of  $\mathbf{A}$  is built out of the SEPs of  $\mathbf{A}$ , according to the Leibniz formula:

$$\det(\mathbf{A}) = \sum_{\ell \in S_n} \text{sgn}(\ell) \prod_{i=1}^n a_{i,\ell_i} \quad (8)$$

The first factor of a SEP could be any entry, say  $a_{1,\ell_1}$ , from the  $n$  entries of the first row of  $\mathbf{A}$ . Taking into account that  $\ell$  is bijective, the factor  $a_{j,\ell_j}$  of a SEP could be any entry from the  $n - j + 1$  entries of the  $j$ th row of  $\mathbf{A}$  satisfying  $\ell_j \neq \ell_1, \ell_j \neq \ell_2, \dots, \ell_j \neq \ell_{j-1}$ .

The factors of a SEP are the nodes of the tree connected with branches, as partly displayed below:



The  $n$ th order lower Hessenberg matrix over  $\mathbb{C}$  is an  $n \times n$  matrix  $\mathbf{H}_n = (h_{i,j})_{1 \leq i,j \leq n}$  whose entries above the superdiagonal, called *trivial*, are all zero. That is,  $h_{i,j} = 0$ , whenever  $j - i > 1$ , as displayed below:

$$\mathbf{H}_n = \begin{pmatrix} h_{1,1} & h_{1,2} & 0 & \dots & 0 & 0 \\ h_{2,1} & h_{2,2} & h_{2,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{n-2,1} & h_{n-2,2} & h_{n-2,3} & \dots & h_{n-2,n-1} & 0 \\ h_{n-1,1} & h_{n-1,2} & h_{n-1,3} & \dots & h_{n-1,n-1} & h_{n-1,n} \\ h_{n,1} & h_{n,2} & h_{n,3} & \dots & h_{n,n-1} & h_{n,n} \end{pmatrix}. \quad (9)$$

$\mathbf{H}_n$  can be considered as the  $n$ th term of the infinite chain of lower Hessenberg matrices

$$\mathbf{H}_1 \sqsubset \mathbf{H}_2 \sqsubset \dots \sqsubset \mathbf{H}_n \sqsubset \dots \quad (10)$$

where the notation  $\mathbf{H}_n \sqsubset \mathbf{H}_{n+1}$  means that  $\mathbf{H}_n$  is a top submatrix of  $\mathbf{H}_{n+1}$ .

The determinant of  $\mathbf{H}_n$  for  $n \geq 2$ , known as *Hessenbergian*, satisfies the recurrence

$$\det(\mathbf{H}_n) = h_{n,n} \det(\mathbf{H}_{n-1}) + \sum_{k=1}^{n-1} \left( (-1)^{n-k} h_{n,k} \prod_{i=k}^{n-1} h_{i,i+1} \det(\mathbf{H}_{k-1}) \right), \quad (11)$$

where  $\det(\mathbf{H}_0) = 1$  and  $\det(\mathbf{H}_1) = h_{1,1}$  (for a proof of the recurrence formula (11) see [11]).

Notice that zero entries (if any) below and including the entries of the superdiagonal are all non-trivial. A SEP of  $\mathbf{H}_n$  will be called *non-trivial* if it exclusively consists of non-trivial entries. Throughout the paper the set of non-trivial SEPs associated with  $\det(\mathbf{H}_n)$  is denoted by  $\mathcal{E}_n$ .

### 3.1 Hessenbergian recurrence in terms of non-trivial SEPs

The non-trivial entries of a Hessenberg matrix  $\mathbf{H}_n$  positioned below and including the main diagonal, namely  $c_{i,j} = h_{i,j}$  for  $j \leq i$ , will be called *standard factors*, while the opposite-sign non-trivial entries in the superdiagonal of  $\mathbf{H}_n$ , namely  $c_{i,i+1} = -h_{i,i+1}$ , will be called *non-standard factors*. We shall also use the alternative notation to  $\mathbf{H}_n$ :

$$\mathbf{H}_n = \begin{pmatrix} c_{1,1} & -c_{1,2} & 0 & \dots & 0 & 0 \\ c_{2,1} & c_{2,2} & -c_{2,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2,1} & c_{n-2,2} & c_{n-2,3} & \dots & -c_{n-2,n-1} & 0 \\ c_{n-1,1} & c_{n-1,2} & c_{n-1,3} & \dots & c_{n-1,n-1} & -c_{n-1,n} \\ c_{n,1} & c_{n,2} & c_{n,3} & \dots & c_{n,n-1} & c_{n,n} \end{pmatrix}. \quad (12)$$

The following Proposition will make clear the usefulness of the matrix form (12).

**Proposition 1.** *i) Let  $C$  be a non-trivial SEP of  $\mathbf{H}_n$  and  $\mathbf{c}$  be the number of non-standard factors of  $C$ . The expansion of (11), when written in terms of SEPs, comprises entirely non-trivial SEPs. Moreover any arbitrary non-trivial SEP in this expansion, say  $C = \text{sgn}(\ell) h_{1,\ell_1} h_{2,\ell_2} \dots h_{n,\ell_n}$ , satisfies  $\text{sgn}(\ell) = (-1)^{\mathbf{c}}$ , that is*

$$C = c_{1,\ell_1} c_{2,\ell_2} \dots c_{n,\ell_n}. \quad (13)$$

*ii) The number of non-trivial SEPs of  $\det(\mathbf{H}_n)$  is  $2^{n-1}$ .*

*Proof.* *i)* Writing (11) in terms of entries of (12) it takes the form:

$$\begin{aligned} \det(\mathbf{H}_n) &= c_{n,n} \det(\mathbf{H}_{n-1}) + \sum_{k=1}^{n-1} \left( (-1)^{n-k} c_{n,k} \prod_{i=k}^{n-1} (-1) c_{i,i+1} \det(\mathbf{H}_{k-1}) \right) \\ &= c_{n,n} \det(\mathbf{H}_{n-1}) + \sum_{k=1}^{n-1} \left( (-1)^{n-k} c_{n,k} (-1)^{n-k} \prod_{i=k}^{n-1} c_{i,i+1} \det(\mathbf{H}_{k-1}) \right), \end{aligned}$$

or equivalently

$$\det(\mathbf{H}_n) = c_{n,n} \det(\mathbf{H}_{n-1}) + \sum_{k=1}^{n-1} \prod_{i=k}^{n-1} c_{n,k} c_{i,i+1} \det(\mathbf{H}_{k-1}).$$

Taking into account that  $\det(\mathbf{H}_0) = 1$  and  $\det(\mathbf{H}_1) = c_{1,1}$ , the latter expression of  $\det(\mathbf{H}_n)$  can be written as:

$$\begin{aligned} \det(\mathbf{H}_n) &= c_{n,1} c_{1,2} c_{2,3} \dots c_{n-1,n} + c_{n,2} c_{2,3} \dots c_{n-1,n} c_{1,1} + c_{n,3} c_{3,4} \dots c_{n-1,n} \det(\mathbf{H}_2) \\ &\quad + \dots + c_{n,n-1} c_{n-1,n} \det(\mathbf{H}_{n-2}) + c_{n,n} \det(\mathbf{H}_{n-1}) \end{aligned} \quad (14)$$

We apply induction on  $n \geq 2$ . Since  $\det(\mathbf{H}_2) = c_{1,1} c_{2,2} - c_{2,1} (-c_{1,2}) = c_{1,1} c_{2,2} + c_{2,1} c_{1,2}$ , the statement holds for  $n = 2$ . Suppose that all the SEPs of  $\mathbf{H}_k$  for  $k \leq n-1$  are non-trivial in the form:  $c_{1,\ell_1} c_{2,\ell_2} \dots c_{i,\ell_i} \dots c_{k,\ell_k}$ . Applying this hypothesis to the right hand side of (14), we infer that every SEP is non-trivial in the form (13), as being product of non-trivial factors. This completes the induction.

*ii)* Let  $\gamma(n)$  be the number of non-trivial SEPs of  $\mathbf{H}_n$ . We apply the induction on  $n \geq 2$ . As

$\gamma(2) = 2^{2-1} = 2$  the statement holds for  $n = 2$ . Suppose that the statement holds for  $k \leq n - 1$ . Then (14) implies that:  $\gamma(n) = 1 + 1 + \gamma(2) + \gamma(3) + \dots + \gamma(n-2) + \gamma(n-1) = 1 + 1 + 2 + 2^2 + \dots + 2^{n-2} = 2^{n-1}$ . This completes the induction.  $\square$

In view of Proposition 1, the determinant in (11) consists of  $\text{card}(\mathcal{E}_n) = 2^{n-1}$  non-trivial SEPs exclusively, while the formula (8) yields the surplus  $n! - 2^{n-1}$  trivial SEPs.

### 3.2 Anatomy of non-trivial SEPs

In this subsection we examine the structure of the non-trivial SEPs, as sequences of standard and non-standard factors.

**Proposition 2.** *Every non-trivial factor associated with  $\mathbf{H}_n$  is a factor of a non-trivial SEP of  $\mathbf{H}_n$ .*

*Proof.* The proof is by induction on  $n \geq 2$ . As  $\det(\mathbf{H}_2) = c_{1,1}c_{2,2} + c_{2,1}c_{1,2}$ , the statement holds for  $n = 2$ . Let  $\mathbf{H}_{n-1}$  fulfil the statement. An inspection of (14) shows that all entries of the  $n$ th row of  $\mathbf{H}_n$  as well as the opposite-sign entries of the superdiagonal (non-standard factors) of  $\mathbf{H}_n$  are factors of non-trivial SEPs of  $\mathbf{H}_n$ . The remaining (standard) factors of SEPs of  $\mathbf{H}_n$  are entries of  $\mathbf{H}_{n-1}$ . The induction entails that they are factors of SEPs of  $\mathbf{H}_{n-1}$ . As shown in (14), all the factors of SEPs of  $\mathbf{H}_{n-1}$  are also factors of SEPs of  $\mathbf{H}_n$  yielded by the product  $c_{n,n} \det(\mathbf{H}_{n-1})$ . This completes the induction.  $\square$

Propositions 1 and 2 justify the terminology “standard and non-standard factors” adopted herein. Notice that Proposition 2 is not true for determinants of either lower or upper triangular matrices, in which we identify as trivial factors the zero entries in the, respectively, upper-left or lower-right corner of the matrix. Such determinants are built out of one non-trivial SEP consisting of the entries of the main diagonal exclusively. In all that follows we adhere to the conventions:  $c_{0,0} = h_{0,0} = 1$  and  $\ell_0 = 0$ .

**Proposition 3.** *Let  $C = c_{1,\ell_1}c_{2,\ell_2}\dots c_{n,\ell_n}$  be a non-trivial SEP of  $\mathbf{H}_n$ . If  $i, k \in \mathbb{Z}^+$  such that  $0 \leq k < i \leq n$ , then:*

$$i + 1 > \ell_k. \quad (15)$$

*Proof.* As  $C$  is non-trivial, all the factors of  $C$  are non-trivial. The definition of a non-trivial factor, say  $c_{k,\ell_k}$ , entails that  $\ell_k - k \leq 1$ , whence  $\ell_k \leq k + 1$ . The hypothesis implies that  $k < i$ , whence  $k + 1 \leq i$ . The assertion follows from:  $\ell_k \leq k + 1 \leq i < i + 1$ .  $\square$

Let  $C = c_{1,\ell_1}\dots c_{k,\ell_k}\dots c_{i-1,\ell_{i-1}}c_{i,\ell_i}\dots c_{n,\ell_n}$  be a non-trivial SEP of  $\mathbf{H}_n$ . A product

$$C[k, i] \stackrel{\text{def}}{=} c_{k,\ell_k}c_{k+1,\ell_{k+1}}\dots c_{i-1,\ell_{i-1}}c_{i,\ell_i}$$

of consecutive factors of  $C$  is said to be a *string*. The string  $C[k, i]$  is said to be a *substring* of  $C[q, p]$  if both  $C[k, i]$  and  $C[q, p]$  are strings of the same non-trivial SEP and satisfy  $q \leq k$  and  $p \leq i$ . In this case, the string  $C[q, p]$  is said to be a *superstring* of  $C[k, i]$ . If  $i = k$ , then  $C[k, k] = c_{k,\ell_k}$ . The string  $C[1, i]$  will be called *initial string* determined by  $i$  and it will be simply denoted by  $C[i]$ . The class of all initial strings determined by  $i$  is denoted by  $\mathfrak{C}[i]$ . Evidently  $C[0] = c_{0,0} = 1$ ,  $\mathfrak{C}[0] = \{c_{0,0}\}$  and  $\mathfrak{C}[1] = \{c_{1,1}, c_{1,2}\}$ . Notice that we can also write  $\mathfrak{C}[1] = \{c_{0,0}c_{1,1}, c_{0,0}c_{1,2}\}$ . Moreover,  $\mathfrak{C}[2] = \{c_{1,1}c_{2,2}, c_{1,1}c_{2,3}, c_{1,2}c_{2,1}, c_{1,2}c_{2,3}\}$ . Even though the initial string  $c_{1,1}c_{2,3}$  (which is included in the SEP  $c_{1,1}c_{2,3}c_{3,2}$ ) is not a SEP, every non-trivial SEP is an initial string, since  $C \in \mathcal{E}_n$  is included in itself. In view of (10), a SEP of  $\mathbf{H}_n$  is not a SEP of  $\mathbf{H}_m$ , whenever  $m \neq n$ . Unlike SEPs, the string  $C[k, n]$  is also a string of  $\mathbf{H}_m$  for all  $m > n$ .

A non-trivial factor  $c_{i,j}$  is called *immediate successor* (IS) of  $C[k, i-1] = c_{k,\ell_k}\dots c_{i-1,\ell_{i-1}}$ ,  $i \geq 1$ , if  $c_{k,\ell_k}\dots c_{i-1,\ell_{i-1}}c_{i,j}$  is a string. By virtue of Proposition 2, every non-trivial factor  $c_{i,j}$ ,  $1 \leq i \leq n$ , associated with  $\mathbf{H}_n$  is an IS of some initial string in  $\mathfrak{C}[i-1]$ . A necessary and sufficient condition for a non-trivial factor to be an IS of an initial string is given below:

**Proposition 4.** *Let  $i \geq 1$  and  $C[i-1] = c_{1,\ell_1}c_{2,\ell_2}\dots c_{i-1,\ell_{i-1}}$  be an initial string in  $\mathfrak{C}[i-1]$ . A non-trivial factor  $c_{i,j}$  of  $\mathbf{H}_n$  is an IS of  $C[i-1]$  if and only if  $j \neq \ell_1, j \neq \ell_2, \dots, j \neq \ell_{i-1}$ .*

*Proof.* If  $i = 1$ , the factors  $c_{1,1}$  and  $c_{1,2}$  are the ISs of  $C[0]$ , since both  $j = 1$  and  $j = 2$  differ from  $\ell_0 = 0$ . If  $i > 1$  and  $c_{i,j}$  is an IS of  $C[i-1]$ , then, by definition, there is  $n \in \mathbb{Z}^+$  and a non-trivial SEP, say  $B$ , of the form  $B = c_{1,\ell_1} c_{2,\ell_2} \dots c_{i-1,\ell_{i-1}} c_{i,\ell_i} \dots c_{n,\ell_n}$  with  $j = \ell_i$ . As  $\ell$  is bijective the result follows.

For the converse statement we assume that  $j \neq \ell_1, j \neq \ell_2, \dots, j \neq \ell_{i-1}$ . First, we construct a non-trivial SEP of  $\mathbf{H}_n$ , which includes  $C[i] = c_{1,\ell_1} c_{2,\ell_2} \dots c_{i-1,\ell_{i-1}} c_{i,j}$ . We define:  $\ell_i = j$ ,  $\ell_{i+1} = i+2, \dots, \ell_{n-1} = n$ . If  $1 \leq r \leq n-i-1$ , then  $i+1 \leq i+r \leq n-1$ . In view of (15), we have:  $\ell_{i+r} = (i+r)+1 > \ell_k$  for all  $k : 0 \leq k < i+r$ . Thus,  $\ell_p \neq \ell_q$  if and only if  $p \neq q$ , provided that  $1 \leq p \leq n-1$  and  $1 \leq q \leq n-1$ . Since  $\{1, 2, \dots, n\} \setminus \{\ell_1, \ell_2, \dots, \ell_{n-1}\}$  is a singleton, say  $\{m\}$ , we define  $\ell_n = m$ . Accordingly, a bijection  $\ell : 1 \mapsto \ell_1, 2 \mapsto \ell_2, \dots, i \mapsto \ell_i, \dots, n \mapsto \ell_n$  has been constructed, which determines a non-trivial SEP including  $C[i]$ . As  $C[i]$  includes  $C[i-1]$ , it follows that  $c_{i,j}$  is an IS of  $C[i-1]$ , as claimed.  $\square$

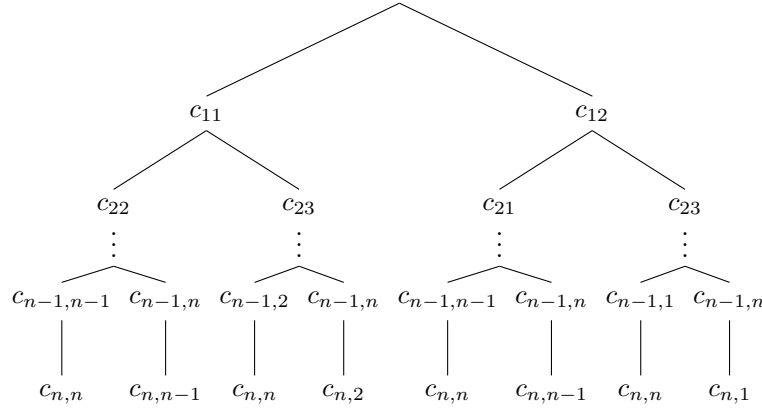
The above Proposition is in accord with the construction of SEPs for complete square matrices, since all the entries of these matrices are non-trivial.

**Proposition 5.** *i) Let  $1 \leq i \leq n-1$ . There are two distinct ISs of  $C[i-1]$  in  $\mathfrak{C}[i-1]$ . One of these ISs is the non-standard factor  $c_{i,i+1}$ .  
ii) Let  $i = n$ . There is only one IS of  $C[n-1]$ , which is standard.*

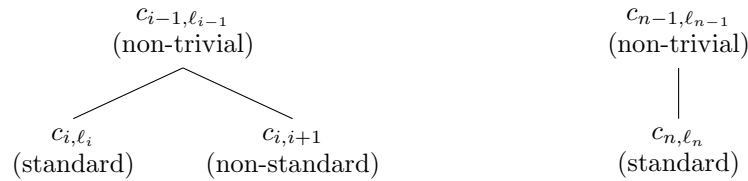
*Proof.* i) Let  $c_{i,j}$  be an IS of  $C[i-1] = c_{1,\ell_1} c_{2,\ell_2} \dots c_{i-1,\ell_{i-1}}$ . In view of Proposition 4 the number of all possible ISs of  $C[i-1]$  coincides with the number of the non-trivial entries of the  $i$ th row of  $\mathbf{H}_n$  satisfying  $j \neq \ell_1, j \neq \ell_2, \dots, j \neq \ell_{i-1}$ . The  $i$ th row contains  $i+1$  non-trivial entries, and therefore there are 2 (as determined by  $i+1-i-1=2$ ) ISs of  $C[i-1]$ , as asserted. It follows from (15) that  $i+1 \neq \ell_{i-1}, \dots, i+1 \neq \ell_1$ . Proposition 4 (applied with  $j = i+1$ ) implies that the non-standard factor  $c_{i,i+1}$  is an IS of  $C[i-1]$ , as claimed.

ii) If  $i = n$ , then the ISs of  $C[n-1]$  are entries of the  $n$ th row, which contains  $n$  non-trivial entries. Thus there is 1 (as determined by  $n - (n-1) = 1$ ) available IS of  $c_{n-1,\ell_{n-1}}$ , which is standard as being entry of the last row.  $\square$

All factors of a non-trivial SEP, say  $C = c_{1,\ell_1} c_{2,\ell_2} \dots c_{n-1,\ell_{n-1}} c_{n,\ell_n}$ , in the tree representation of  $\det(\mathbf{H}_n)$  are nodes, from which two branches start, up to factor  $c_{n-1,\ell_{n-1}}$ , from which only one branch starts. These results are partly displayed below:



Furthermore, at each node is rooted one branch ending at a non-standard factor and another ending at a standard factor. All branches rooted at node  $c_{n-1,\ell_{n-1}}$  end at a standard factor. The results are portrayed in the following figures:



The fact that the number of non-trivial SEPs of  $\mathbf{H}_n$  is  $2^{n-1}$  is thus re-established. Evidently  $\text{card}(\mathfrak{C}[n]) = 2^n$ . The standard ISs of strings are classified below.

**Proposition 6.** *i) Let  $1 \leq i \leq n$ . The standard IS of any standard factor  $c_{i-1, \ell_{i-1}}$  is  $c_{i, i}$ .  
ii) If  $2 \leq i \leq n$  and the factors of  $C[i-1]$  are all non-standard, then the standard IS of  $C[i-1]$  is  $c_{i, 1}$ .  
iii) Let  $C[i-k-1, i-1]$  be a string such that  $3 \leq i \leq n$  and  $1 \leq k \leq i-2$ . If the first factor of  $C[i-k-1, i-1]$  is standard and the substring  $C[i-k, i-1]$  of  $C[i-k-1, i-1]$  consists exclusively of non-standard factors, that is*

$$C[i-k, i-1] = \underbrace{c_{i-k, i-k+1} c_{i-k+1, i-k+2} \dots c_{i-1, i}}_k,$$

*then the standard IS of  $C[i-k-1, i-1]$  is  $c_{i, i-k}$ .*

*Proof.* i) Let us consider an arbitrary initial string  $C[i-1]$ , which contains the standard factor  $c_{i-1, \ell_{i-1}}$  (ending factor). In view of Proposition 4, we need to verify that none of the factors of  $C[i-1]$  has as column position the index  $i$ . As the entries  $c_{k, i}$  for  $1 \leq k \leq i-2$  are all trivial, it follows from (15) that the index  $i$ , as being the column position of  $c_{i, i}$ , is not a column index of the predecessors of  $c_{i-1, \ell_{i-1}}$  in  $C[i-1]$ . As  $c_{i-1, \ell_{i-1}}$  is standard, we further infer that  $i \neq \ell_{i-1}$  and the result follows.

ii) The hypothesis entails that the predecessors of  $c_{i, i}$  are the non-standard factors:  $c_{1, 2}, \dots, c_{i-1, i}$ . As  $\ell_i = 1$  is not a column index of these predecessors the assertion follows.

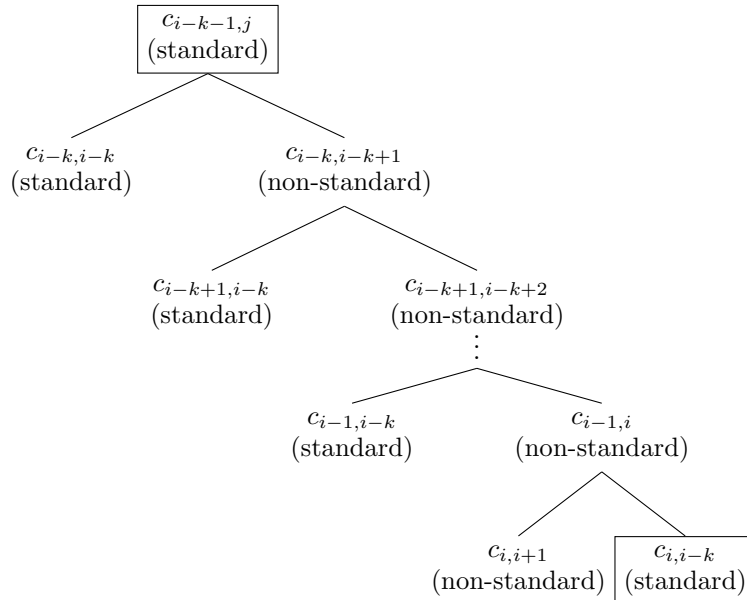
iii) Let us consider any initial string,  $C[i-1]$ , which includes  $C[i-k-1, i-1]$ , that is

$$C[i-1] = c_{1, \ell_1} \dots c_{i-k-2, \ell_{i-k-2}} c_{i-k-1, \ell_{i-k-1}} c_{i-k, i-k+1} c_{i-k+1, i-k+2} \dots c_{i-1, i},$$

where  $c_{1, \ell_1}, \dots, c_{i-k-2, \ell_{i-k-2}}$  are arbitrary. We need to verify that none of the factors of  $C[i-1]$  has as column position the index  $i-k$ . As  $c_{i-k-1, \ell_{i-k-1}}$  is standard, it follows that  $\ell_{i-k-1} \neq i-k$ . Evidently,  $i-k$  is not a column index of  $c_{i-k, i-k+1}, \dots, c_{i-2, i-1}, c_{i-1, i}$ . Finally, on account of (15),  $i-k$  is not a column index of the predecessors of  $c_{i-k-1, \ell_{i-k-1}}$ , since all these predecessors are trivial.  $\square$

The results of Proposition 6 can be rephrased as follows: The standard IS of a standard factor is the entry of the main diagonal in the successor row (statement 1). The standard factor whose predecessors are  $k$  consecutive non-standard factors is  $c_{i, i-k}$  (statement 3). As special case, if  $k = i-1$ , then all the predecessors of  $c_{i, \ell_i}$  are non-standard factors, whence  $c_{i, \ell_i} = c_{i, 1}$  (statement 2).

The above results are illustrated in the following sub-tree of the tree representation of  $\det(\mathbf{H}_n)$ :





## 4 Non-trivial SEPs as arrays of 0s & 1s

The results of the previous section are applied herein to represent each non-trivial SEP by a finite array of 0s and 1s in one-to-one fashion. It provides in (17) an alternative expression to the recurrence (11) leading closer to the desired expression in (26).

### 4.1 The representation theorem

In the rest of this paper  $\mathcal{Z}^n$  will stand for the set of functions from  $\{1, 2, \dots, n\}$  to  $\{0, 1\}$ , that is  $\mathcal{Z}^n = \{(r_1, r_2, \dots, r_{n-1}, r_n) : r_i = 0 \text{ or } 1\}$ . The set  $\mathcal{Z}^n$  can be identified with the segment of the non-negative binary integers up to and including the binary integer  $2^n - 1$  (see section 5). The set  $\mathfrak{R}_n$  is defined as the subset of  $\mathcal{Z}^n$  consisting of the elements of  $\mathcal{Z}^n$  whose last component is  $r_n = 1$ , that is:  $\mathfrak{R}_n = \{r \in \mathcal{Z}^n : r_n = 1\}$ . Evidently  $\text{card}(\mathcal{Z}^n) = 2^n$  and  $\text{card}(\mathfrak{R}_n) = 2^{n-1}$ . An element  $r \in \mathfrak{R}_n$  will be denoted as  $r = (r_1, r_2, \dots, r_{n-1}, 1)$ . In the following definition we introduce a simple rule for associating arrays in  $r \in \mathfrak{R}_n$  with non-trivial SEPs in  $\mathcal{E}_n$ .

**Definition 1.** We define the function  $f^{(n)} : \mathcal{E}_n \ni C = c_{1,\ell_1} c_{2,\ell_2} \dots c_{n,\ell_n} \mapsto f^{(n)}(C) \in \mathcal{Z}^n$  by:

$$f^{(n)}(C) \stackrel{\text{def}}{=} (r_1, r_2, \dots, r_{n-1}, r_n) : r_i = \begin{cases} 0, & \text{if } \ell_i = i + 1 \\ 1, & \text{if } \ell_i \neq i + 1 \end{cases} \quad (16)$$

That is, every  $C = c_{1,\ell_1} c_{2,\ell_2} \dots c_{n,\ell_n} \in \mathcal{E}_n$  is mapped through  $f^{(n)}$  to  $r = (r_1, r_2, \dots, r_{n-1}, r_n) \in \mathcal{Z}^n$ , according to the rule:  $r_i = 0$ , whenever  $c_{i,\ell_i}$  is non-standard or  $r_i = 1$ , whenever  $c_{i,\ell_i}$  is standard.

As the elements of the last row are all standard factors, the last component of  $f^{(n)}(C)$  is 1, that is:  $f^{(n)}(C) \in \mathfrak{R}_n$ . Therefore,  $f^{(n)} : \mathcal{E}_n \ni C \mapsto f^{(n)}(C) \in \mathfrak{R}_n$ . The assumption  $r_i = 0$  entails that  $i \neq n$ , that is:  $1 \leq i \leq n - 1$ .

**Theorem 1 (Representation).** The function  $f^{(n)} : \mathcal{E}_n \mapsto \mathfrak{R}_n$  in Definition 1 is bijective.

*Proof.* Since the set  $\mathfrak{R}_n$  and the set  $\mathcal{E}_n$  have the same number of elements ( $2^{n-1}$ ) it suffices to show that  $f^{(n)}$  is injective. Let us consider  $C = c_{1,\ell_1} c_{2,\ell_2} \dots c_{n,\ell_n}$  and  $P = c_{1,l_1} c_{2,l_2} \dots c_{n,l_n}$  in  $\mathcal{E}_n$  such that  $f^{(n)}(C) = f^{(n)}(P)$ . We need to show that  $C = P$  or equivalently that  $\ell = l$ . Let us call  $r = f^{(n)}(C) = f^{(n)}(P)$  and  $r = (r_1, r_2, \dots, r_{n-1}, 1)$ . We examine the following cases:

I) If  $r_i = 0$ , then, by Definition 1, the  $i$ th non-trivial factors both of  $C$  and  $P$  are non-standard, whence they coincide with the factor  $c_{i,i+1}$ . Thus  $\ell_i = l_i = i + 1$ .

II) If  $r_i = 1$ , then Definition 1 implies that the  $i$ th factors of  $C$  and  $P$  are standard. In this case, there are three possible subcases:

IIa)  $r_i = r_{i-1} = 1$  and  $2 \leq i \leq n$ .

It follows from Definition 1 that the factors  $c_{i-1,\ell_{i-1}}$ ,  $c_{i,\ell_i}$  and  $c_{i-1,l_{i-1}}$ ,  $c_{i,l_i}$  are all standard. By virtue of Proposition 6 statement (i) the IS of any standard factor is the unique factor  $c_{i,i}$ . Thus  $\ell_i = l_i = i$ .

IIb)  $r_i = r_{i-1-k} = 1$  and  $r_m = 0$  for all  $m$  such that  $i - k \leq m \leq i - 1$ , whenever  $3 \leq i \leq n$  and  $1 \leq k \leq i - 2$ .

It follows from Definition 1 that the factors  $c_{i-1-k,\ell_{i-1-k}}$ ,  $c_{i,\ell_i}$  and  $c_{i-1-k,l_{i-1-k}}$ ,  $c_{i,l_i}$  are standard, while the strings  $S_1 = c_{i-k,\ell_{i-k}}, \dots, c_{i-1,\ell_{i-1}}$  and  $S_2 = c_{i-k,l_{i-k}}, \dots, c_{i-1,l_{i-1}}$  consist entirely of non-standard factors. Proposition 6 statement (ii) implies that the ISs of  $S_1$  and  $S_2$  coincide with the factor  $c_{i,i-k}$ , whence  $\ell_i = l_i = i - k$ .

IIc)  $r_i = 1$  and  $r_m = 0$  for all  $m = 1, 2, \dots, i - 1$ .

It follows from Definition 1 that  $c_{i,\ell_i}$  and  $c_{i,l_i}$  are standard, while the initial strings  $S_1 = c_{1,\ell_1}, \dots, c_{i-1,\ell_{i-1}}$  and  $S_2 = c_{1,l_1}, \dots, c_{i-1,l_{i-1}}$  consist entirely of non-standard factors. Proposition 6 statement (iii) implies that the ISs of  $S_1$  and of  $S_2$  coincide with  $c_{i,1}$ , whence  $\ell_i = l_i = 1$ .

The proof of the Theorem is complete.  $\square$

As an illustrative example, consider the non-trivial SEP:  $T = c_{1,1}c_{2,3}c_{3,2}c_{4,5}\dots c_{n-2,n-1}c_{n-1,4}c_{n,n} \in \mathcal{E}_n$  for  $n \geq 8$ .  $T$  is represented by the array  $r = (1, 0, 1, 0, 0, \dots, 0, 1, 1) \in \mathfrak{R}_n$ , that is  $f^{(n)}(T) = r$ . Next, we verify that the inverse image of  $r$  is  $T$ , that is  $(f^{(n)})^{-1}(r) = T$ . By Definition 16 the non-standard factors occupy the same positions as the 0s in  $r$ , that is, the positions  $i = 2, 4, 5, \dots, n-2$  are occupied by the non-standard factors  $c_{2,3}, c_{4,5}, c_{5,6}, \dots, c_{n-2,n-1}$ , respectively. Since  $r_1 = 1$ , it follows that the (unique) standard factor of  $T$  is  $c_{1,1}$ . As  $r_1 = 1, r_2 = 0, r_3 = 1$ , Proposition 6 (ii) entails that the third factor of  $T$  is  $c_{3,3-1} = c_{3,2}$ . By analogy, as the number of consecutive 0s between  $r_3 = 1$  and  $r_{n-1} = 1$  is  $k = n-5$ , on account of  $n-1 - (n-5) = 4$ , the  $(n-1)$ th factor of  $T$  is  $c_{n-1,4}$ . As  $r_n = 1$  and  $r_{n-1} = 1$ , Proposition 6 (i) entails that  $k=0$ , and therefore the last factor of  $T$  is  $c_{n,n}$ , as expected.

## 4.2 An intermediate Hessenbergian expression

Throughout this paper  $\varphi^{(n)}$  stands for the inverse function of  $f^{(n)}$ , that is  $\varphi^{(n)} = (f^{(n)})^{-1}$ . Moreover in the determinant expansion formulas each SEP represents its product value:  $\prod_{i=1}^n c_{i,\ell_i}$ . Taking into account that the determinant expansion of  $\mathbf{H}_n$  in (12) is the sum of all non-trivial SEP product values, Theorem 1 entails that every term in the sum of  $\det(\mathbf{H}_n) = \sum_{C \in \mathcal{E}_n} C$  can be replaced by  $\varphi^{(n)}(r)$ ,  $r \in \mathfrak{R}_n$ , that is

$$\det(\mathbf{H}_n) = \sum_{r \in \mathfrak{R}_n} \varphi^{(n)}(r). \quad (17)$$

The expression in (17) consists entirely of  $\text{card}(\mathfrak{R}_n) = 2^{n-1}$  distinct non-trivial SEPs. The disadvantage of this formula is related to the fact that the sum variable ranges over the set of arrays in  $\mathfrak{R}_n$ .

## 5 Hessenbergian closed form via elementary integer functions

In this section we introduce a suitable function which associates integers from  $\mathbb{I}_{n-1} = \{0, 1, \dots, 2^{n-1} - 1\}$  with arrays in  $\mathfrak{R}_n$  in one-to-one fashion. This will enable us to replace the indexing set  $\mathfrak{R}_n$  in (17) with  $\mathbb{I}_{n-1}$ ,  $n \in \mathbb{Z}^+$ , leading to the closed form of  $\det(\mathbf{H}_n)$ .

Throughout the paper,  $\mathcal{B}_n$  denotes the set of binary integers from 0 up to and including the number

$$\mathbf{1}_n = \underbrace{11\dots 1}_n \quad (n \text{ number of } 1\text{s})$$

that is  $\mathcal{B}_n = \{0, 1, 10, \dots, \mathbf{1}_n\}$ . Evidently  $\mathcal{B}_n$  consists of  $2^n$  binary numbers. The binary representation of the integer  $2^n - 1$  is  $\mathbf{1}_n$ , that is  $[2^n - 1]_2 = \mathbf{1}_n$ .

Let  $b \in \mathcal{B}_n$  with  $b \neq 0$ . By completing the binary figures of  $b = 1r_{k+1}\dots r_n$  by  $k-1$  zero digits at its left up to and including the binary figure  $2^{n-1}$ , we adhere to the standard conventions

$$\begin{array}{ccccccc} 1r_{k+1}\dots r_n \equiv & 0 & 0 & \dots & 0 & 1 & r_{k+1} \dots r_n & \text{and} & \mathbf{0}_n \equiv & 0 & 0 & \dots & 0 \\ & \uparrow & & & & \uparrow & \uparrow & & \uparrow & & & & \uparrow \\ \text{Binary Figures :} & 2^{n-1} & & & & 2^{n-k} & 2^0 \text{ (units)} & & 2^{n-1} & & & & 2^0 \end{array}$$

which lead to the identification of the elements of  $\mathcal{Z}^n$  with  $\mathcal{B}_n$ . For example, if  $n = 5$ , then the binary integer  $11 \in \mathcal{B}_5$  is identified with the array  $(0, 0, 0, 1, 1) \in \mathcal{Z}^5$ .

Taking into account that  $\text{card}(\mathcal{B}_{n-1}) = \text{card}(\mathfrak{R}_n) = 2^{n-1}$ , we define the bijection  $\rho^{(n)} : \mathcal{B}_{n-1} \mapsto \mathfrak{R}_n$ :

$$\rho^{(n)}(\underbrace{00\dots 01r_{k+1}\dots r_{n-1}}_{n-1}) = (\underbrace{0, 0, \dots, 0, 1, r_{k+1}, \dots, r_{n-1}}_n, 1) \quad \text{and} \quad \rho^{(n)}(\underbrace{00\dots 0}_{n-1}) = (\underbrace{0, 0, \dots, 0}_n, 1) \quad (18)$$

By identifying  $\mathcal{B}_{n-1}$  with  $\mathfrak{R}_n$  through  $\rho^{(n)}$ , the function  $\varphi^{(n)}$  defined above associates every binary integer  $r$  in  $\mathcal{B}_{n-1}$  with the SEP  $\varphi^{(n)}(r)$ .

## 5.1 Nested divisions

Let  $\kappa \in \mathbb{Z}^*$  and  $\lambda \in \mathbb{Z}^+$ . The largest integer not greater than the rational number  $\kappa/\lambda$ , will be denoted as  $\lfloor \kappa/\lambda \rfloor$ . Also  $\lceil \kappa/\lambda \rceil$  denotes the smallest integer not less than  $\kappa/\lambda$ . The notation  $\lfloor \kappa/\lambda \rfloor$  coincides with the quotient of the Euclidean division of  $\kappa$  by  $\lambda$ , also known as *integral part* (or integer part) of the number  $\kappa/\lambda$ . We adopt the method of converting an integer  $m \in \mathbb{I}_{n-1}$  into a binary number  $[m]_2 = r_1 r_2 \dots r_{n-1} \in \mathcal{B}_{n-1}$  based on the Euclidean division. The digits  $r_i$  in  $[m]_2$  are the remainders of nested divisions:

$$m = 2q_{n-1} + r_{n-1}, q_{n-1} = 2q_{n-2} + r_{n-2}, \dots, q_2 = 2q_1 + r_1.$$

Taking into account that

$$q_{n-1} = \lfloor m : 2 \rfloor, q_{n-2} = \lfloor \lfloor m : 2 \rfloor : 2 \rfloor, \dots, q_1 = \lfloor \underbrace{\lfloor \dots \lfloor m : 2 \rfloor : 2 \rfloor \dots}_{n-1} : 2 \rfloor$$

the  $r_i$ s in  $[m]_2$  can be expressed in terms of the greatest integer function “ $\lfloor \cdot \rfloor$ ” and of the modulo function:

$$r_{n-1} = m \bmod 2, r_{n-2} = \lfloor m : 2 \rfloor \bmod 2, \dots, r_1 = \lfloor \underbrace{\lfloor \dots \lfloor m : 2 \rfloor : 2 \rfloor \dots}_{n-2} : 2 \rfloor \bmod 2 \quad (19)$$

We can also write  $r_{n-1} = \lfloor m : 2^0 \rfloor \bmod 2$ , which leads to the unified expression

$$r_i = \lfloor \underbrace{\lfloor \dots \lfloor m : 2 \rfloor : 2 \rfloor \dots}_{n-i-1} : 2 \rfloor \bmod 2,$$

where  $1 \leq i \leq n-1$ .

In the following Proposition we provide a condensed expression for nested divisions.

**Proposition 7.** *Let  $m \in \mathbb{Z}$ . The following identity of nested divisions holds:*

$$\lfloor \underbrace{\lfloor \dots \lfloor m : 2 \rfloor : 2 \rfloor \dots}_{k} : 2 \rfloor = \lfloor m : 2^k \rfloor. \quad (20)$$

*Proof.* Let  $x$  be a real number and  $p, q$  be positive integers. We shall use the well known identity

$$\lfloor \lfloor x \rfloor : p \rfloor = \lfloor x : p \rfloor. \quad (21)$$

Taking into account that  $(x : q) : p = x : (p \cdot q)$ , it follows from (21) that:

$$\lfloor \lfloor x : q \rfloor : p \rfloor = \lfloor (x : q) : p \rfloor = \lfloor x : (p \cdot q) \rfloor. \quad (22)$$

To verify (20) we use induction on  $k \in \mathbb{Z}^*$ . As  $m = \lfloor m : 2^0 \rfloor$  the identity holds for  $k = 0$ . Let us assume that the identity (20) holds for  $k = n$ , that is:

$$\lfloor \underbrace{\lfloor \dots \lfloor m : 2 \rfloor : 2 \rfloor \dots}_{n} : 2 \rfloor = \lfloor m : 2^n \rfloor.$$

In view of (22) we have

$$\lfloor \underbrace{\lfloor \dots \lfloor m : 2 \rfloor : 2 \rfloor \dots}_{\lfloor m : 2^n \rfloor} : 2 \rfloor = \lfloor \lfloor m : 2^n \rfloor : 2 \rfloor = \lfloor m : 2^{n+1} \rfloor,$$

which completes the induction. □

## 5.2 The main result

The binary equivalent  $[m]_2 = r_1 r_2 \dots r_{n-1} \in \mathcal{B}_{n-1}$  of  $m \in \mathbb{I}_{n-1}$  can be expressed, as described in (19) and (20), in terms of elementary integer functions as follows:

$$[m]_2 = (\lfloor m : 2^{n-2} \rfloor \bmod 2, \lfloor m : 2^{n-3} \rfloor \bmod 2, \dots, \lfloor m : 2^0 \rfloor \bmod 2). \quad (23)$$

The relation (23) induces the bijective transformation:

$$\beta^{(n)} : \mathbb{I}_{n-1} \ni m \mapsto \beta^{(n)}(m) = [m]_2 \in \mathcal{B}_{n-1}. \quad (24)$$

The composite  $\tau^{(n)} \stackrel{\text{def}}{=} \rho^{(n)} \circ \beta^{(n)}$  determines a bijection, which converts non-negative integers into arrays in  $\mathfrak{R}_n$ :

$$\tau^{(n)}(m) = (\lfloor m : 2^{n-2} \rfloor \bmod 2, \lfloor m : 2^{n-3} \rfloor \bmod 2, \dots, \lfloor m : 2^0 \rfloor \bmod 2, 1). \quad (25)$$

Moreover, for every  $n \in \mathbb{N}$  and every  $m \in \mathbb{N}$  such that  $m < 2^{n-1}$ :

$$[\tau^{(n)}(m)]_{10} = 2m + 1.$$

Finally, the composition of  $\varphi^{(n)}$  (introduced in section 4) and  $\tau^{(n)}$  yields the bijection  $\chi^{(n)} \stackrel{\text{def}}{=} \varphi^{(n)} \circ \tau^{(n)}$ :

$$\begin{array}{ccc} \mathbb{I}_{n-1} & \xrightarrow{\tau^{(n)}} & \mathfrak{R}_n \\ \chi^{(n)} & \searrow \downarrow & \varphi^{(n)} \\ & \mathcal{E}_n & \end{array}$$

The composite  $\chi^{(n)}$  associates integers from  $\mathbb{I}_{n-1}$  to complex numbers (the product values of the SEPs) and is defined once  $\mathbf{H}_n$  is given. The results of this section enable us to modify (17) in order to reach a closed form (our initial quest) of  $\det(\mathbf{H}_n)$  as will be described in the following Theorem.

**Theorem 2.** *The closed form of  $\det(\mathbf{H}_n)$  is:*

$$\det(\mathbf{H}_n) = \sum_{m=0}^{2^{n-1}-1} \chi^{(n)}(m). \quad (26)$$

*Proof.* As  $\tau^{(n)} : \mathbb{I}_{n-1} \ni m \mapsto \tau^{(n)}(m) \in \mathfrak{R}_n$  is bijective, every  $r \in \mathfrak{R}_n$  can be replaced in (17) by  $\tau^{(n)}(m)$ ,  $m \in \mathbb{I}_{n-1}$ . Taking into account that  $\chi^{(n)}$  is bijective, (17) takes the form

$$\det(\mathbf{H}_n) = \sum_{r \in \mathfrak{R}_n} \varphi^{(n)}(r) = \sum_{m \in \mathbb{I}_{n-1}} \varphi^{(n)}(\tau^{(n)}(m)) = \sum_{m=0}^{2^{n-1}-1} \chi^{(n)}(m),$$

as required.  $\square$

## Examples

To illustrate the closed form of  $\det(\mathbf{H}_n)$  in (26), we consider the Hessenbergians of order:  $n = 2, 3, 4$ .

The expansion of  $\det(\mathbf{H}_2)$  consists of  $2^1$  non-trivial SEPs, and  $\mathbb{I}_1 = \{0, 1\}$ . In view of (25), the arrays  $\tau^{(2)}(m) \in \mathfrak{R}_2$  are given by:

$$\begin{aligned} \tau^{(2)}(0) &= (\lfloor 0 : 2^{2-2} \rfloor \bmod 2, 1) = (0 \bmod 2, 1) = (0, 1) \\ \tau^{(2)}(1) &= (\lfloor 1 : 2^{2-2} \rfloor \bmod 2, 1) = (1 \bmod 2, 1) = (1, 1). \end{aligned}$$

Recalling that the non-standard factors are opposite-sign entries of the  $\mathbf{H}_n$  superdiagonal, the non-trivial SEPs of  $\mathbf{H}_2$  are:

$$\chi^{(2)}(0) = \varphi^{(2)}(\tau^{(2)}(0)) = \varphi^{(2)}(0, 1) = -h_{1,2}h_{2,1}, \quad \chi^{(2)}(1) = \varphi^{(2)}(\tau^{(2)}(1)) = \varphi^{(2)}(1, 1) = h_{1,1}h_{2,2}.$$

Thus,

$$\det(\mathbf{H}_2) = \sum_{m=0}^1 \chi^{(2)}(m) = h_{1,1}h_{2,2} - h_{1,2}h_{2,1}.$$

The expansion of  $\det(\mathbf{H}_3)$  consists of  $2^2$  non-trivial SEPs, and  $\mathbb{I}_2 = \{0, 1, 2, 3\}$ . The arrays  $\tau^{(3)}(m) \in \mathfrak{R}_3$  are given by:

$$\begin{aligned} \tau^{(3)}(0) &= ( \lfloor 0 : 2^{3-2} \rfloor \bmod 2, \lfloor 0 : 2^{3-3} \rfloor \bmod 2, 1) \\ &= ( 0 \bmod 2, 0 \bmod 2, 1) \\ &= ( 0, 0, 1) \\ \tau^{(3)}(1) &= ( \lfloor 1 : 2^{3-2} \rfloor \bmod 2, \lfloor 1 : 2^{3-3} \rfloor \bmod 2, 1) \\ &= ( 0 \bmod 2, 1 \bmod 2, 1) \\ &= ( 0, 1, 1) \\ \tau^{(3)}(2) &= ( \lfloor 2 : 2^{3-2} \rfloor \bmod 2, \lfloor 2 : 2^{3-3} \rfloor \bmod 2, 1) \\ &= ( 1 \bmod 2, 2 \bmod 2, 1) \\ &= ( 1, 0, 1) \\ \tau^{(3)}(3) &= ( \lfloor 3 : 2^{3-2} \rfloor \bmod 2, \lfloor 3 : 2^{3-3} \rfloor \bmod 2, 1) \\ &= ( 1 \bmod 2, 3 \bmod 2, 1) \\ &= ( 1, 1, 1) \end{aligned}$$

The non-trivial SEPs of  $\mathbf{H}_3$  are listed below:

$$\begin{aligned} \chi^{(3)}(0) &= \varphi^{(3)}(\tau^{(3)}(0)) = \varphi^{(3)}(0, 0, 1) = -h_{1,2}(-h_{2,3})h_{3,1} = h_{1,2}h_{2,3}h_{3,1} \\ \chi^{(3)}(1) &= \varphi^{(3)}(\tau^{(3)}(1)) = \varphi^{(3)}(0, 1, 1) = -h_{1,2}h_{2,1}h_{3,3} \\ \chi^{(3)}(2) &= \varphi^{(3)}(\tau^{(3)}(2)) = \varphi^{(3)}(1, 0, 1) = h_{1,1}(-h_{2,3})h_{3,2} = -h_{1,1}h_{2,3}h_{3,2} \\ \chi^{(3)}(3) &= \varphi^{(3)}(\tau^{(3)}(3)) = \varphi^{(3)}(1, 1, 1) = h_{1,1}h_{2,2}h_{3,3} \end{aligned}$$

Thus,

$$\det(\mathbf{H}_3) = \sum_{m=0}^3 \chi^{(3)}(m) = h_{1,2}h_{2,3}h_{3,1} - h_{1,2}h_{2,1}h_{3,3} - h_{1,1}h_{2,3}h_{3,2} + h_{1,1}h_{2,2}h_{3,3}.$$

Let us finally consider the  $\det(\mathbf{H}_4)$ . It consists of  $2^3$  non-trivial SEPs, and  $\mathbb{I}_3 = \{0, 1, \dots, 7\}$ . The arrays  $\tau^{(4)}(m) \in \mathfrak{R}_4$  are given by:

$$\begin{aligned} \tau^{(4)}(0) &= ( \lfloor 0 : 2^{4-2} \rfloor \bmod 2, \lfloor 0 : 2^{4-3} \rfloor \bmod 2, \lfloor 0 : 2^{4-4} \rfloor \bmod 2, 1) \\ &= ( 0 \bmod 2, 0 \bmod 2, 0 \bmod 2, 1) \\ &= ( 0, 0, 0, 1) \\ \tau^{(4)}(1) &= ( \lfloor 1 : 2^{4-2} \rfloor \bmod 2, \lfloor 1 : 2^{4-3} \rfloor \bmod 2, \lfloor 1 : 2^{4-4} \rfloor \bmod 2, 1) \\ &= ( 0 \bmod 2, 0 \bmod 2, 1 \bmod 2, 1) \\ &= ( 0, 0, 1, 1) \\ \tau^{(4)}(2) &= ( \lfloor 2 : 2^{4-2} \rfloor \bmod 2, \lfloor 2 : 2^{4-3} \rfloor \bmod 2, \lfloor 2 : 2^{4-4} \rfloor \bmod 2, 1) \\ &= ( 0 \bmod 2, 1 \bmod 2, 2 \bmod 2, 1) \\ &= ( 0, 1, 0, 1) \\ \tau^{(4)}(3) &= ( \lfloor 3 : 2^{4-2} \rfloor \bmod 2, \lfloor 3 : 2^{4-3} \rfloor \bmod 2, \lfloor 3 : 2^{4-4} \rfloor \bmod 2, 1) \\ &= ( 0 \bmod 2, 1 \bmod 2, 3 \bmod 2, 1) \\ &= ( 0, 1, 1, 1) \\ \tau^{(4)}(4) &= ( \lfloor 4 : 2^{4-2} \rfloor \bmod 2, \lfloor 4 : 2^{4-3} \rfloor \bmod 2, \lfloor 4 : 2^{4-4} \rfloor \bmod 2, 1) \\ &= ( 1 \bmod 2, 2 \bmod 2, 4 \bmod 2, 1) \\ &= ( 1, 0, 0, 1) \end{aligned}$$

$$\begin{aligned}
\tau^{(4)}(5) &= ( \lfloor 5 : 2^{4-2} \rfloor \bmod 2, \quad \lfloor 5 : 2^{4-3} \rfloor \bmod 2, \quad \lfloor 5 : 2^{4-4} \rfloor \bmod 2, \quad 1) \\
&= ( \quad 1 \bmod 2, \quad \quad 2 \bmod 2, \quad \quad 5 \bmod 2, \quad 1) \\
&= ( \quad \quad 1, \quad \quad \quad 0, \quad \quad \quad 1, \quad \quad 1) \\
\tau^{(4)}(6) &= ( \lfloor 6 : 2^{4-2} \rfloor \bmod 2, \quad \lfloor 6 : 2^{4-3} \rfloor \bmod 2, \quad \lfloor 6 : 2^{4-4} \rfloor \bmod 2, \quad 1) \\
&= ( \quad 1 \bmod 2, \quad \quad 3 \bmod 2, \quad \quad 6 \bmod 2, \quad 1) \\
&= ( \quad \quad 1, \quad \quad \quad 1, \quad \quad \quad 0, \quad \quad 1) \\
\tau^{(4)}(7) &= ( \lfloor 7 : 2^{4-2} \rfloor \bmod 2, \quad \lfloor 7 : 2^{4-3} \rfloor \bmod 2, \quad \lfloor 7 : 2^{4-4} \rfloor \bmod 2, \quad 1) \\
&= ( \quad 1 \bmod 2, \quad \quad 3 \bmod 2, \quad \quad 7 \bmod 2, \quad 1) \\
&= ( \quad \quad 1, \quad \quad \quad 1, \quad \quad \quad 1, \quad \quad 1)
\end{aligned}$$

The non-trivial SEPs of  $\mathbf{H}_4$  are listed below:

$$\begin{aligned}
\chi^{(4)}(0) &= \varphi^{(4)}(\tau^{(4)}(0)) = \varphi^{(4)}(0, 0, 0, 1) = -h_{1,2}(-h_{2,3})(-h_{3,4})h_{4,1} = -h_{1,2}h_{2,3}h_{3,4}h_{4,1} \\
\chi^{(4)}(1) &= \varphi^{(4)}(\tau^{(4)}(1)) = \varphi^{(4)}(0, 0, 1, 1) = -h_{1,2}(-h_{2,3})h_{3,1}h_{4,4} = h_{1,2}h_{2,3}h_{3,1}h_{4,4} \\
\chi^{(4)}(2) &= \varphi^{(4)}(\tau^{(4)}(2)) = \varphi^{(4)}(0, 1, 0, 1) = -h_{1,2}h_{2,1}(-h_{3,4})h_{4,3} = h_{1,2}h_{2,1}h_{3,4}h_{4,3} \\
\chi^{(4)}(3) &= \varphi^{(4)}(\tau^{(4)}(3)) = \varphi^{(4)}(0, 1, 1, 1) = -h_{1,2}h_{2,1}h_{3,3}h_{4,4} \\
\chi^{(4)}(4) &= \varphi^{(4)}(\tau^{(4)}(4)) = \varphi^{(4)}(1, 0, 0, 1) = h_{1,1}(-h_{2,3})(-h_{3,4})h_{4,2} = h_{1,1}h_{2,3}h_{3,4}h_{4,2} \\
\chi^{(4)}(5) &= \varphi^{(4)}(\tau^{(4)}(5)) = \varphi^{(4)}(1, 0, 1, 1) = h_{1,1}(-h_{2,3})h_{3,2}h_{4,4} = -h_{1,1}h_{2,3}h_{3,2}h_{4,4} \\
\chi^{(4)}(6) &= \varphi^{(4)}(\tau^{(4)}(6)) = \varphi^{(4)}(1, 1, 0, 1) = h_{1,1}h_{2,2}(-h_{3,4})h_{4,3} = -h_{1,1}h_{2,2}h_{3,4}h_{4,3} \\
\chi^{(4)}(7) &= \varphi^{(4)}(\tau^{(4)}(7)) = \varphi^{(4)}(1, 1, 1, 1) = h_{1,1}h_{2,2}h_{3,3}h_{4,4}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\det(\mathbf{H}_4) &= \sum_{m=0}^7 \chi^{(4)}(m) = -h_{1,2}h_{2,3}h_{3,4}h_{4,1} + h_{1,2}h_{2,3}h_{3,1}h_{4,4} + h_{1,2}h_{2,1}h_{3,4}h_{4,3} - h_{1,2}h_{2,1}h_{3,3}h_{4,4} \\
&\quad + h_{1,1}h_{2,3}h_{3,4}h_{4,2} - h_{1,1}h_{2,3}h_{3,2}h_{4,4} - h_{1,1}h_{2,2}h_{3,4}h_{4,3} + h_{1,1}h_{2,2}h_{3,3}h_{4,4}.
\end{aligned}$$

The above results coincide with the determinant expansions derived from the Leibniz formula, yet excluding the trivial SEPs.

## 6 The general solution of regular order LDEVCs

We will finally describe the solution expressions in all three types of regular order LDEVCs (ascending-order,  $N$ -order and unbounded-order), derived from the closed form of Hessenbergians.

As the solution matrices  $\Xi_n^{(i)}$ ,  $\mathbf{P}_n$ ,  $\mathbf{G}_n$  associated with the ascending order LDEVC (see section 2) are all in lower Hessenberg form, the determinants of the fundamental, particular and general solution matrices are all Hessenbergians. As a consequence, the formula (26) is directly applicable to each solution determinant.

More specifically, the general solution of the ascending order LDEVC, is obtained by identifying the general solution matrix  $\mathbf{G}_n$  in (6) with the matrix:

$$\mathbf{H}_{n+1} = \begin{pmatrix} h_{1,1} & h_{1,2} & 0 & \dots & 0 & 0 \\ h_{2,1} & h_{2,2} & h_{2,3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{n,1} & h_{n,2} & h_{n,3} & \dots & h_{n,n} & h_{n,n+1} \\ h_{n+1,1} & h_{n+1,2} & h_{n+1,3} & \dots & h_{n+1,n} & h_{n+1,n+1} \end{pmatrix}. \quad (27)$$

In other words, the entries  $h_{i,j}$  of  $\mathbf{H}_{n+1}$  are assigned with the values:

$$h_{i,j} = \begin{cases} g_{i-1} - \sum_{k=0}^{N-1} a_{i-1,k} y_{k-N} & \text{if } 1 \leq i \leq n+1 \text{ and } j = 1, \\ a_{i-1,N+j-2} & \text{if } 1 \leq i \leq n+1 \text{ and } j = 2, \dots, i+1, \\ 0 & \text{otherwise.} \end{cases}$$

Applying the closed form (26) to  $\mathbf{H}_{n+1}$ , the general solution of the ascending order LDEVC in (7) takes the following (closed) form:

$$y_n = (-1)^n \frac{\sum_{m=0}^{2^n-1} \chi^{(n+1)}(m)}{\prod_{i=0}^n a_{i,i+N}}. \quad (28)$$

We can further reduce (28) by changing the 1st column entries of the general solution matrix (27). In particular, we assign  $\mathbf{H}_{n+1} = (h_{i,j})_{1 \leq i,j \leq n+1}$  (or  $\mathbf{G}_n$ ) with the entries:

$$h_{i,j} = \begin{cases} (-1)^n \frac{g_{i-1} - \sum_{k=0}^{N-1} a_{i-1,k} y_{k-N}}{\prod_{i=0}^n a_{i,i+N}} & \text{if } 1 \leq i \leq n+1 \text{ and } j = 1, \\ a_{i-1,N+j-2} & \text{if } 1 \leq i \leq n+1 \text{ and } j = 2, \dots, i+1, \\ 0 & \text{otherwise} \end{cases} \quad (29)$$

Applying the multilinear property of determinants with respect to the 1st column of  $\mathbf{H}_{n+1}$ , as defined in (29), the general solution described in (7) can be expressed as a single Hessenbergian:

$$y_n = \det(\mathbf{H}_{n+1}). \quad (30)$$

Finally, the expression (26), applied to (30), gives the ascending order LDEVC general solution a more condensed, alternative to (28), closed form:

$$y_n = \sum_{m=0}^{2^n-1} \chi^{(n+1)}(m). \quad (31)$$

The  $N$ th order LDEVC is also associated with the solution matrices (fundamental, particular and general)  $\mathbf{\Xi}_n^{(i)}$ ,  $\mathbf{P}_n$  and  $\mathbf{G}_n$ , respectively. Its general solution is represented by the formulas (28) (or (31)). In this case, however, the solution matrices are even more sparse. In particular, the fundamental solution matrix  $\mathbf{\Xi}_n^{(i)}$  is a band matrix in which additional zero entries are grouped at its bottom left corner. This produces additional zero, but in our terminology non-trivial, SEPs that are also included in the formula.

In the unbounded order case (where  $N = 0$ ), homogeneous solutions do not exist. Thus  $\mathbf{G}_n = \mathbf{P}_n$  and the unique solution of the unbounded order LDEVC is derived from the closed form expression of the Hessenbergian  $\det(\mathbf{P}_n)$  in (4).

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